Hodge Theory Lecture 3

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Cohomology

Recall last time the de Rham cohomology: the set of closed forms which are not exact. This is very vague, and so we will define it now:

Definition 0.1. Given a smooth manifold, the de Rham cohomology group is the set

$$\text{Closed Forms}/\sim$$

where two closed forms are equivalent iff they differ by an exact form. More precisely,

$$H^k_{dR}(M) = \{\omega \in \Omega^k(M) : d\omega = 0\}/\{\alpha \in \Omega^k(M) : \alpha = d\beta\}$$

This can be seen as fairly natural, since two closed forms that differ by an exact form represent the same "obstruction".

An example of the de Rham cohomology of a space is $\mathbb{R}^2 - \{0\}$. The 1-dimensional classes are $\mathbb{R}$-linear combinations of $[0], [d\theta]$, though $c[0] = 0[d\theta] = [0]$ and hence this is just $\mathbb{R}$-linear combinations of $[d\theta]$. Thus

$$H^1_{dR}(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}$$

Some of the most important facts are summarized as follows:

Theorem 0.1. *de Rham cohomology (assuming the manifold is compact) satisfies the following:*

1. If $F : M \to N$ is smooth, then the pullback $F^* : H^k_{dR}(N) \to H^k_{dR}(M)$ is linear.

2. If $F : M \to N$ is a diffeomorphism, then $H^k_{dR}(M) \cong H^k_{dR}(N)$ for each k

3. If $\dim(M) = m$, then $H^k_{dR}(M) = 0$ for $k < 0$ or $k > m$

4. $H^0_{dR}(M) \cong \mathbb{R}^N$ where N is the number of connected components

5. If F, G are smoothly homotopic, then the induced maps of cohomology groups are the same

6. The Mayer-Vietoris principle(see Lee)

Much of this holds true for general manifolds, but is technically more difficult, and not useful in Hodge theory. For the proof see Lee.

Obstructions

Consider the following fact: On a star-shaped open subset(of $\mathbb{R}^n$), every closed form is exact. That is, if $d\omega = 0$ then $\omega = d\alpha$. We also have the same for every open set diffeomorphic to a star-shaped open subset. Thus

Theorem 0.2. *Given a smooth manifold M and a point $x \in M$, then there exists $U \ni x$ open such that every closed form on U is exact.*

Proof. Every point has a neighbourhood diffeomorphic to an open ball □

Thus we have locally every cohomology group is trivial, but we already found an example of a manifold with non-trivial cohomology. Thus, we deduce that there must be an obstruction to piecing together local solutions to global ones.

Let us explore this a bit more.

Suppose that $M = \bigcup_j U_j$ where U_j is diffeomorphic to a star-shaped open subset and every intersection of two of them is connected. Then as with smooth functions, if

$$\alpha_i \in \Omega^{k-1}(U_i) \text{ and } \alpha_i - \alpha_j = 0 \text{ on } U_i \cap U_j$$

then $\alpha(x) = \alpha_i(x), x \in U_i$ is a differential form on the whole manifold. Thus if $d\omega = 0$ and $d\alpha_i = \omega|_{U_i}$, then for ω to be exact it is sufficient for $\alpha_i - \alpha_j = 0$ on $U_i \cap U_j$.

On the other hand let us suppose that $k = 1$, suppose that $\omega = d\beta$, then $d(\alpha_i - \beta) = 0$ so that $\alpha_i - \beta = c_i = const.$.

Hence it is necessary for there to exist constants c_i such that $(\alpha_i + c_i) - (\alpha_j + c_j) = 0$, and in this case then $\beta = \{\alpha_i + c_i\}$ is well-defined with $d\beta = \omega$.

Finally, note that $\alpha_i - \alpha_i = const = K_{ij}$ on $U_i \cap U_j$. So once we pick α_i we need to pick appropriate c_i so that $c_i - c_j = K_{ij}$.

Thus we have

Theorem 0.3. *Suppose that $M = \bigcup_j U_j$ where U_j is diffeomorphic to a star-shaped open subset and every intersection of two of them is connected. Then a closed form ω is exact iff $\exists \alpha_i \in \Omega^0(U_i)$ with $d\alpha_i = \omega|_{U_i}$ and constants c_i such that*

$$c_i - c_j = K_{ij}$$

Two properties of K_{ij} we should note, in general: $K_{ij} = -K_{ji}$ and $0 =$

$K_{ik} + K_{kj} + K_{ji}$ since

$$\alpha_i - \alpha_k + \alpha_k - \alpha_j + \alpha_j - \alpha_i = 0$$

In particular, every closed form yields a (non-unique) 1-cocycle, and is exact iff this cocycle is a coboundary.

Exercise 0.1. Check that regardless of our choice of α_i satisfying the requirements, the iff still holds. In particular, one choice is a coboundary iff any other one is. Also, two such cocycles(choices of K_{ij}) differ by a coboundary.

While we have assumed ω is a 1-form, some analogous conditions holds for higher forms.

Symbols

This section will be fairly short.

Suppose that we have two vector bundles E, F equipped with smoothly varying inner products at each point(on a Riemannian manifold we could take any tensor bundle). Suppose that the base manifold M is Riemannian. Then given an operator

$$P : \Gamma(E) \to \Gamma(F)$$

there exists an operator $P^* : \Gamma(F) \to \Gamma(E)$ given by

$$\int_M \langle Pf, g \rangle_F \, dvol = \int_M \langle f, P^* g \rangle_F \, dvol$$

If P is a differential operator then so is P^*, and similarly the degree is the same, and ellipticity is preserved. Furthermore, the principal symbols $\sigma(P) = \sigma(P^*)^T$, the transpose.(NOTE: THIS DEPENDS ON THE INNER PRODUCTS AND ONLY HOLDS POINTWISE)

Now, suppose that $P = d$, then it can be shown that $\sigma(d)(x,\xi)\beta = \xi \wedge \beta(x)$, and $\sigma(d^*)(x,\xi)\beta = \iota_\xi \beta(x)$, and hence

$$\sigma(dd^* + dd^*)(x,\xi) = (\xi \wedge *) \circ \iota_\xi + \iota_\xi \circ (\xi \wedge *) = |\xi|^2$$

We see immediately that $dd^* + dd^*$ is elliptic. We may use that

$$\sigma(dd^* + dd^*) = (\sigma(d + d^*))^2$$

to also conclude the first order operator $d + d^* : \Omega^*(M) \to \Omega^*(M)$ is elliptic.

Exercise 0.2. Prove this by working in local coordinates. What is $|\xi|^2$?

One final thing to note is the following: If P is first-order elliptic, then

$$\|u\|_{H^1} \leq C(\|u\|_{L^2} + \|Pu\|_{L^2})$$

thus with a possibly different constant

$$\frac{1}{C}(\|u\|_{L^2} + \|du\|_{L^2} + \|d^*u\|_{L^2}) \leq \|u\|_{H^1} \leq C(\|u\|_{L^2} + \|du\|_{L^2} + \|d^*u\|_{L^2})$$